String-like behaviour of 4d SU(3) Yang-Mills flux tubes

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ABSTRACT: We present here results on the fine structure of the static $q\bar{q}$ potential in $d = 4$ SU(3) Yang-Mills theory. The potential is obtained from Polyakov loop correlators having separations between 0.3 and 1.2 fermi. Measurements were carried out on lattices of spatial extents of about 4 and 5.4 fermi. The temporal extent was 5.4 fermi in both cases. The results are analyzed in terms of the force between a $q\bar{q}$ pair as well as in terms of a scaled second derivative of the potential. The data is accurate enough to distinguish between different effective string models and it seems to favour the expression for ground state energy of a Nambu-Goto string.

Contents

1. Introduction

The string picture of hadrons $[1-3]$, has a long history starting from the 1960's. Although this picture could explain the Regge trajectories of hadrons, it required 26 space-time dimensions to be a consistent theory [4]. Effective string theories which could be quantized in any dimensions were introduced by Polchinski and Strominger in [5]. In QCD such strings have the interpretation of gluonic flux tubes between a quark and an antiquark. Nambu made formal connections between QCD and the string models [6].

A long distance $1/r$ term in the quark-antiquark potential, distinct from the short distance couloumbic $1/r$ term was first observed in [7]. Subsequently Lüscher showed the universality of this term and since then it is known as the Lüscher term [8], having the value $-\pi(d-2)/24r$ where d is the number of space time dimensions and r the length of the string.

In the eighties [9] and [10] claimed to find the Lüscher term in $d = 3$ SU(2) and $d = 3 + 1$ SU(3) lattice gauge theories respectively. However precise data on large Wilson loops or Polyakov loop correlators, necessary to identify the class of the effective string did not exist at that time.

In recent times, significant progress has been made due to increase in computing power as well as improvement in algorithms. See $[11-16]$ for example.

Related to the effective string descriptions is the issue of the spectrum of string excitations. Recently analytical studies, based only on symmetry principles have been carried out in [17, 18]. Extensive numerical studies have been carried out in [19] and [13, 14]. We do not discuss this topic here. One can refer to the review article by Kuti [20] for a general introduction and more details.

In this article we present results of our simulations of the Polyakov loop correlators for $d = 4$ SU(3) Yang-Mills theory and compare the resulting static potential with both perturbation theory and string model predictions. A preliminary report of this simulation was presented in [21].

2. c(r) at short and long distances

In this article we look at $c(r) = \frac{1}{2} r^3 F'(r)$ where $F(r)$ is the force between a quark and an antiquark. This quantity asymptotically tends to the Lüscher term for large r. We measure $c(r)$ at intermediate distances from slightly below 0.5 fermi to about 1.2 fermi on a 32^4 lattice at $\beta = 5.7$. It is therefore interesting to compare the values one obtains from perturbation theory at short distances as well as from the string picture.

$c(r)$ can be easily computed from 2-loop perturbation theory using the results in [22] and [23]. We just quote the final result for the sake of completeness:

$$c(r) = -C_F \left[\left\{ \alpha_{\overline{\mathrm{MS}}} + c_0 (\alpha_{\overline{\mathrm{MS}}})^2 \right\} - \frac{r}{2} \partial_r \alpha_{\overline{\mathrm{MS}}} \left\{ 1 + 2 c_0 \alpha_{\overline{\mathrm{MS}}} \right\} \right] + \cdots . \tag{2.1}$$

Here $C_F = \frac{4}{3}$ and $c_0 = 8\pi\beta_0 \left(\gamma - \frac{35}{66} \right)$. The function $\alpha_{\overline{\mathrm{MS}}}$ is given by [23]

$$\alpha_{\overline{\mathrm{MS}}}(r) = \frac{1}{4\pi\beta_0 f_1} \left(1 - \frac{\beta_1 f_2}{\beta_0^2 f_1} \right) \tag{2.2}$$

with $f_1 = -\ln \left(r^2 \Lambda_{\overline{\mathrm{MS}}}^2 \right)$ and $f_2 = \ln (f_1)$. For SU(3) the constants β_0, β_1 and γ have their usual continuum values of $\frac{11}{(4\pi)^2}$, $\frac{102}{(4\pi)^4}$ and 0.57722 respectively. The result of this computation is plotted as the curve '2-loop perturbation theory' in fig 2 with the upper and lower curves corresponding to estimated upper and lower limits of $\Lambda_{\overline{\mathrm{MS}}}$ [22].

At the other end of the length scale one expects a string like behaviour. Since non-bosonic forms of the string have been ruled out [24], our case of interest is the Nambu-Goto string. The potential in this case was first given by Arvis [25] to be

$$V_{\mathrm{Arvis}}(r) = \sqrt{\sigma^2 r^2 - \frac{(d-2)\pi}{12}\sigma}. \tag{2.3}$$

An interesting point, noted even before Arvis is that the $\bar{q}q$ potential becomes purely imaginary for $r < r_c$ where $r_c = \sqrt{\frac{(d-2)\pi}{12\sigma}}$ [26], which is around 0.3 fermi. This behaviour was connected to the tachyon instability of the Nambu-Goto string [27]. It is curious that perturbative calculations also break down at about the same distance.

Another form of the potential that is of interest is the so called truncated Arvis potential which is obtained by expanding the Arvis potential in a power series and retaining the first three terms:

$$V_{\mathrm{trunc}} = \sigma r - \frac{(d-2)\pi}{24r} - \frac{(d-2)^2 \pi^2}{1152 \sigma r^3}. \tag{2.4}$$

3. Simulation

3.1 Algorithm

Since we are interested in ground state properties, we measure Polyakov loop correlators as it is well known that the Polyakov loop correlator for large temporal extent strongly projects onto the $q\bar{q}$ ground state. We use the multilevel technique of Lüscher and Weisz for measuring the Polyakov loop correlators very accurately. For details of the algorithm we refer the reader to [12].

This algorithm has several optimization parameters and the most important among them seems to be the number of "measurements" used to compute certain intermediate expectation values. We will refer to this number henceforth as "iupd". Another parameter is the thickness of the time slice. For our case a thickness of two was optimal.

It is also well known that improved observables can be constructed by replacing a bare link by its group average keeping the environment unchanged. This is known as multihit and we employ that too on the time-like links for the correlators. The group average is computed through the single-link integral defined by

$$\langle U \rangle = Z(J, J^\dagger)^{-1} \frac{\partial Z(J, J^\dagger)}{\partial J^\dagger} \quad \text{where} \quad Z(J, J^\dagger) = \int_{\text{SU}(3)} [dU] e^{\text{tr}(UJ^\dagger + U^\dagger J)}. \tag{3.1}$$

For SU(3) the group average cannot be easily carried out analytically and is most often evaluated using a monte-carlo method. An alternative semi-analytic method for this averaging was proposed by de Forcrand and Roiesnel [28]. We used the semi-analytic method for the multihit and we estimate that this resulted in a 60% speedup of the code compared to using the monte-carlo method to reach similar levels of accuracy.

In our test runs, for Monte-Carlo multihit the optimal value for iupd occurred at the same place for the two different values of r we considered but not for the semi-analytic case. To obtain maximum gain from the multilevel scheme, we tried to choose iupd close to the optimal values for our largest r. Another interesting observation is that the penalty paid for operating at non-optimal values of iupd seems to be higher for Monte-Carlo multihit than the semi-analytic method.

We have carried out simulations at $\beta = 5.7$ on both $24^3 \times 32$ and 32^4 lattices using the Wilson gauge action. The lattice spacing at this β is 0.17 fm so that the temporal extent of the lattice is 5.4 fm while the spatial box is $(4 \, \text{fm})^3$ in one case and $(5.4 \, \text{fm})^3$ in the other. Details about setting the scale is discussed in the next sub-section.

For separations $r = 2 - 6$ each measurement involved simulations on $24^3 \times 32$ lattices with iupd=12000 and 500 measurements were made in all. For the larger separations $r = 5 - 9$ simulations were done on 32^4 lattices with iupd=48000 and in all again about 500 measurements were obtained.

We also found that for the larger separations ($r = 6$ and higher), we had to go to larger lattices to continue to gain from the multilevel scheme. On the $24^3 \times 32$ lattice increasing "iupd" even by an order of magnitude did not seem to help reduce the error on the $r = 6$ correlator. However going to a larger lattice (32^4) did help significantly.

3.2 Results

In our simulations we measure the Polyakov loop correlator $\langle P^* P \rangle(r)$ where r is the separation between the two Polyakov loops. The $q\bar{q}$ potential is determined as

$$V(r) = -\frac{1}{T} \log \langle P^* P \rangle(r) \tag{3.2}$$

where T is the temporal extent of the lattice. In our analysis we look at the force between the $q\bar{q}$-pair given by $F(r) = \frac{dV(r)}{dr}$ and the scaled second derivative $c(r)$ given by $c(r) =$

r	$\langle P^*P\rangle(r)$	$\bar{r}$	force	$\tilde{r}$	$c(\tilde{r})$
2	$6.683(19)\times10^{-12}$				
3	$6.824(38)\times10^{-15}$	2.277	0.21521(9)	2.700	-0.3076(3)
4	$1.894(17)\times10^{-17}$	3.312	0.18396(11)	3.729	-0.3218(8)
5	$7.755(44)\times10^{-20}$	4.359	0.17155(13)	4.786	-0.3141(21)
6	$3.842(30)\times10^{-22}$	5.393	0.16591(6)	5.833	-0.3031(28)
7	$2.098(21)\times10^{-24}$	6.414	0.16286(10)	6.864	-0.302(11)
8	$1.217(17)\times10^{-26}$	7.428	0.16100(20)		
9	$7.40(21)\times10^{-29}$	8.438	0.15975(98)		

Table 1: Results: Polyakov loop, Force, $c(r)$

#	functional form	fit range (r)	s	c	$\chi^2/d.o.f$
Fit 1	$s + c/r^2$	6-9	0.1554(1)	0.304(3)	0.12
Fit 2	$s + c/r^2 + 3c^2/2sr^4$	6-9	0.1559(1)	0.266(3)	0.27
Fit 3	$s(1 - \pi/(6sr^2))^{-\frac{1}{2}}$	7-9	0.1560(1)	–	0.3

Table 2: Fit results

$\frac{r^3}{2}\frac{d^2V(r)}{dr^2}$. The latter tends to the asymptotic value of $-\frac{(d-2)\pi}{24}$ for large r which is nothing but the Lüscher term. We are interested in how this quantity approaches its asymptotic value. On the lattice these quantities are defined by

$$F(\bar{r}) = V(r) - V(r-1) \tag{3.3}$$

$$c(\tilde{r}) = \frac{\tilde{r}^3}{2}[V(r+1) + V(r-1) - 2V(r)] \tag{3.4}$$

where $\bar{r}$ and $\tilde{r}$ are defined as in [29] to reduce lattice artifacts. Our results are presented in table 1.

Since different r values are measured in the same simulation, they are quite strongly correlated. Thus when one evaluates the differences in potentials, it helps if one evaluates the differences for each measurement and then averages over different measurements rather than the other way round. In practice we compute the difference between jackknife bins and then compute the jackknife error for the differences. We also checked that our combination of one heat-bath with three over-relaxation sweeps and restricting the measurement to every fifth update resulted in negligible autocorrelation.

The $q\bar{q}$ potential contains an unphysical constant which masks the properties of the flux tube to a certain degree. We therefore look directly at the force defined as in eq. (3.3). In figure 1 we have plotted the force versus $1/r^2$. We use the force to set the scale on the lattice via $r_0^2 F(r_0) = 1.65$ where r_0 is the Sommer parameter which we take to represent the physical distance of $0.5\,\mathrm{fm}$. In our case the Sommer parameter turns out to be 2.93 lattice spacings.
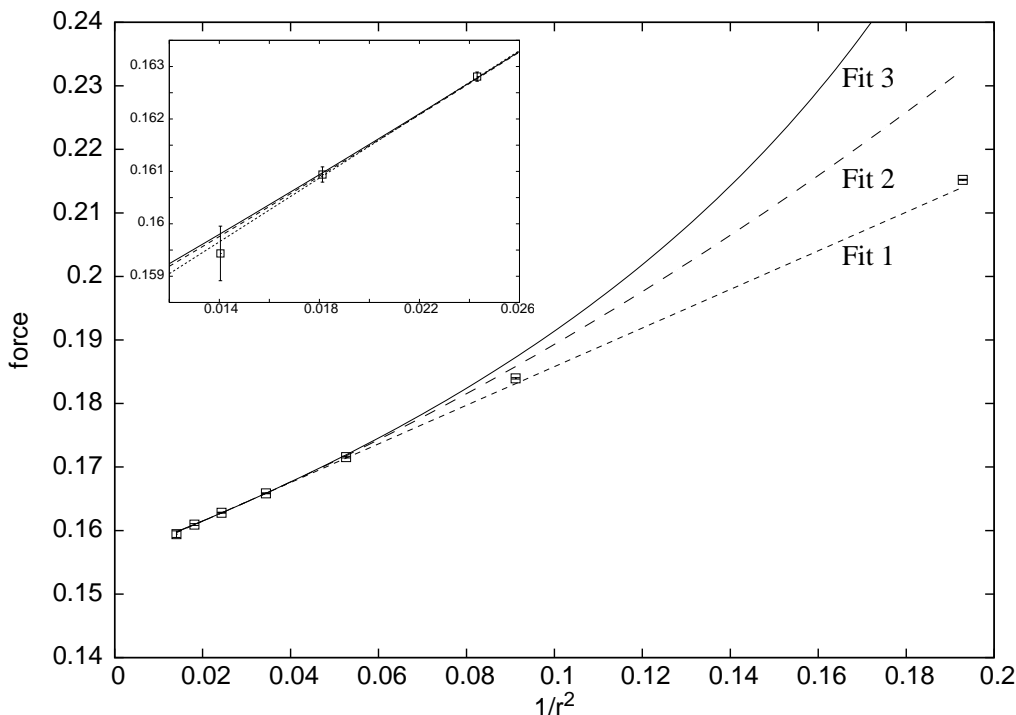
Figure 1: Force in $d = 4$ SU(3) case. The inset shows the last three points in greater detail. The fits 1, 2 and 3 are to a straight line, truncated Arvis and full Arvis respectively. Details about the fits are given in table 2.

We do three different fits to the force data whose results are shown in table 2 . First we fit to the form $sx + c$ where $x = 1/r^2$ with s and c as fit parameters. From the intercept of this fit we obtain the string tension to be $\sigma a^2 = 0.1554(1)$. The slope of this line gives us the effective $c(r)$ over this range of r and this turns out to be $0.304(3)$ which is still 16% away from the asymptotic value of $\pi/12$. This clearly shows that we are still some distance away from the region where the model independent leading order behaviour is all that matters.

Next we fit to the form $s + cx + 3c^2x^2/2s$ which is inspired by the truncated Arvis potential eq. (2.4). This fit gives us the string tension via s as $0.1559(1)$ and the effective $c(r)$ to be $0.266(3)$ which within errors is almost identical to the universal value of $\pi/12$. Finally we fit to the form expected from the full Arvis potential. Within the fit range and errors, this fit is indistinguishable from the fit to the truncated Arvis potential. Unfortunately we cannot yet do an analysis of the the type envisaged in [17] where an additional coefficient in front of the $1/r^4$ term can be determined by fits. Such an analysis would probably require data with significantly reduced error bars. That seems to be out of scope at the moment. What we do see is that the $1/r^3$ term in the potential for open strings with fixed end boundary conditions is indeed consistent with the Arvis potential and any correction to this would have to be really small.

In figure 2 we have plotted the scaled double derivative $c(\tilde{r})$ as a function of r/r_0. The horizontal line at $c(r) = -\pi/12$ is the asymptotic value. Also shown are the predictions of
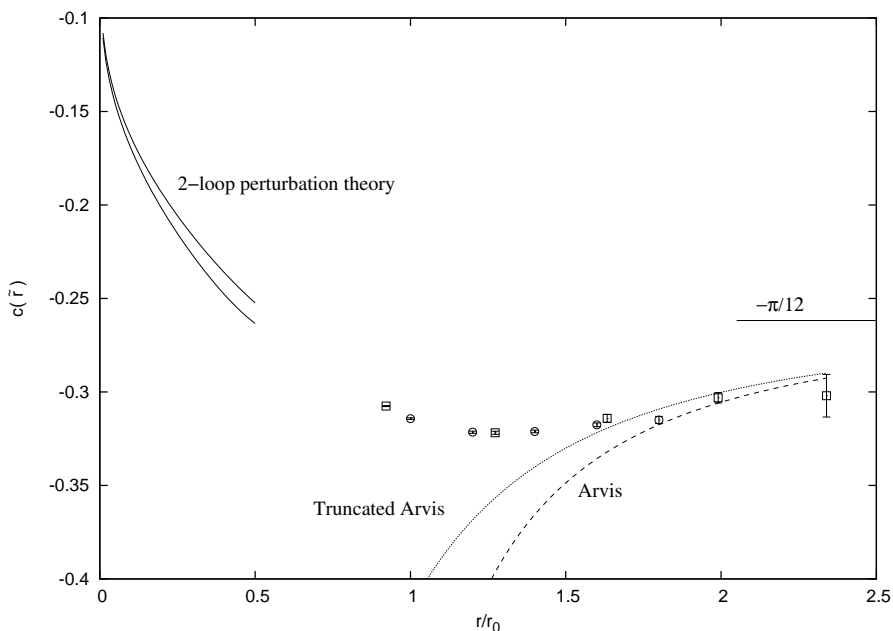
Figure 2: Scaled second derivative for $d = 4$ SU(3) case. The circles are from [29]

$c(r)$ from the Arvis as well as the truncated Arvis potentials. The perturbative predictions are plotted for small values of r extending upto about 0.25 fermi.

4. Discussion and conclusions

In this article we have looked at how the QCD-string behaves at intermediate distances. The only other simulations of the Polyakov loop correlators in $d = 4$ SU(3) theory are reported in [12, 29]. To compare our results with them we have plotted their data along with ours (circles in figure 2). In the range where we overlap, the agreement of the data is excellent. We have been able to extend the range modestly albeit at a considerable computational cost.[1] However even this extension is important as our two extra points definitely show that the data falls quite nicely on the Arvis curve which was not so clear yet with the previous simulations. We unfortunately cannot really distinguish between the truncated Arvis potential and the full Arvis potential as the difference between the two curves is less than 1% already at $2r_0$ or 1 fermi.

Polchinski and Strominger [5] found the Lüscher term as the leading $1/r$ correction in their effective string theory. It has now been claimed (for closed strings) [30 – 32] that even the coefficient of the $1/r^3$ term is the same as the one obtained from the Arvis potential. We find it interesting that the same seems to be true for the open strings also. While in effective string theories there is no apriori reason to expect the full Nambu-Goto behaviour, at the moment we do not see any deviations from it beyond a certain distance.

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[1]In [29] it was estimated that for a given lattice, the computational cost to reach the same level of accuracy in $c(r)$ grows as r^4 with r.

This simulation was performed on a comparatively coarse lattice. Existing results on finer lattices [29] have shown, for the smaller r values, that the data lies above the values obtained in this simulation. If that behaviour persists for larger r values too, then the scale at which convergence with truncated Arvis potential sets in would be larger. It would be really interesting to study the continuum limit of this scale. However that is work for the future.

Acknowledgments

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